# Pseudospin Model for Hard-Core Bosons with Attractive Interaction. Zero Temperature*

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We consider the many-body ground state and elementary excitations of the Matsubara-Matsuda cell model, which can be interpreted as a first approximation to Siegert's exact but untractable operator algebra for hard-core boson fields. The model has unphysical aspects, but allows treatment of some many-body effects of the interaction. It represents the hard core by restricting cell occupation numbers to $\leq 1$, which changes the algebra from Bose to Pauli type; kinetic energy and attraction appear as interactions between nearest-neighbor cell pseudospins, equivalent to a Heisenberg ferromagnet with anisotropy in pseudospin space. Our treatment, valid for scattering length $f_0 \geq 0$, splits the Hamiltonian into an isotropic "unperturbed" part $H_0$, describing a system with hard core plus attraction of strength making $f_0 = 0$, and $AH_1$, consisting of the attraction's deviation from this strength, the parameter $A \geq 0$ giving the magnitude of this (repulsive) deviation. The exact ground state of $H_0$, for any density, is the state symmetric in all pseudospins, with the appropriate eigenvalue of the total pseudospin component $S^{(3)}$ which measures $N$. These states exhibit what corresponds to "incomplete Bose-Einstein condensation," the "excluded volume" effect of the hard-core constraint producing relative depletion $\xi_0$ of the condensate, proportional to $\rho$. Exact single excitations of $H_0$ are density fluctuations $\rho_{\mathbf{k}}$ with, however, free particle-like excitation spectrum, the ground-state energy being density-independent. Relaxing the restriction to eigenstates of $N$ permits definition, by rotation of the total pseudospin from the vacuum, of a quasiparticle vacuum and operators, for any mean density. These serve as starting points for treating the full Hamiltonian by the equations-of-motion method in the random-phase approximation. The excitation spectrum is now phonon-like for small $k$, with $s \sim (\rho A)^{1/2}$. $f_0$ being expressible exactly in terms of $A$, $E_0/N$ can be written in terms of $f_0$, $\rho$, and $\xi_0$, self-consistent in the RPA to order $f_0^{5/2}$. In the low-density limit, $\xi_0 \to 0$, there results the well-known expansion in $(\rho f_0^3)$, but there are higher density corrections including a term $\sim \rho f_0 (\rho f_0^3)^{1/3} \xi_0^{2/3}$, due to the strong interaction included in the *unperturbed* many-body ground state.

## I. INTRODUCTION; NATURE OF THE MODEL

WE consider the ground state and elementary excitations of a simple "pseudospin" model which embodies some characteristics of a many-body Bose system with hard core plus attractive interaction. The model, which has been used previously by Matsubara and Matsuda[1] for treating the λ transition in liquid helium, has some obviously nonphysical features (primarily an artificial "band" structure due to the use of lattice quantization, resulting in anisotropy of and a quasimomentum cutoff in the excitation spectrum). However, it is possible in this model to treat certain aspects of the strong short-range interaction exactly, in zero order, and, for reasons to be outlined presently, we believe that the properties of the model may be of interest for a better understanding of the physical many-body Bose system.

As has been pointed out by Siegert,[2] the presence of a hard-core interaction profoundly modifies the structure of the field description of the many-body Bose system: There is no longer a unitary transformation from a representation in terms of free particle states (or operators) to one in terms of the eigenstates of the interacting system, so that a perturbation treatment starting from the noninteracting system, is not, strictly, possible at all. Of course, for an actual system such as liquid helium, one need not use the idealization of a completely impenetrable core. Nevertheless, the expectation of a profound qualitative effect of the strong short-range repulsion remains. For a *one-dimensional* system of hard-core bosons one knows that the properties are very different from the free-particle (or weak-repulsion) case, the energy spectrum and space correlations being identical with those of the corresponding *fermion* system.[3] In three dimensions the effects of the hard core are probably not quite as drastic. In the low-density limit it was shown by Lee, Huang, and Yang[4] and by Brueckner and Sawada[5] that the hard-core effects may be incorporated into the basic Bogoliubov[6] theory of the weakly nonideal Bose gas by treating the two-particle scattering with sufficient accuracy. However, in contrast to the fermion case where the statistics itself tends to *suppress* the many-particle effects of the short-range repulsion, there is no reason to suppose that in the boson case a pair approximation is at all adequate except in the low-density limit. On the contrary, e.g.,

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[1] T. Matsubara and H. Matsuda, Progr. Theoret. Phys. (Kyoto) **16**, 569 (1956); **17**, 19 (1957).
[2] A. J. F. Siegert, Phys. Rev. **116**, 1057 (1959).
[3] M. Girardeau, J. Math. Phys. **1**, 516 (1960). In the pseudospin model these results follow immediately by applying a Klein transformation [K. Baumann and R. Sexl, Nucl. Phys. **26**, 117 (1961); M. Bolsterli, Phys. Rev. **122**, 1946 (1961)] to the pseudospin operators. R. T. Whitlock, Western Reserve University dissertation, 1963, and T. D. Schultz, J. Math. Phys. **4**, 666 (1963).
[4] T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957).
[5] K. A. Brueckner and K. Sawada, Phys. Rev. **106**, 1117, 1128 (1957).
[6] N. N. Bogoliubov, J. Phys. U.S.S.R. **11**, 23 (1947).

the Green's function treatment of Beliaev[7] shows clearly how the Bose condensation *enhances* the contributions of three-particle (and higher) scattering amplitudes. Thus, it would be highly desirable, if one knew how, to work in a representation which includes the essential many-body effects of the hard core from the start.

Now Siegert's[2] work shows how the hard-core part of the interaction, instead of being treated as the limit of a repulsive potential in the Hamiltonian, may be described by algebraic relations between the field operators which follow from the basic constraint.

$$\psi(\mathbf{r})\psi(\mathbf{r}')=0 \quad \text{for} \quad |\mathbf{r}-\mathbf{r}'|\leq a,$$

where $a$ is the hard-core diameter. These algebraic relations amount to a nonlocal $q$-number modification of the basic commutator $[\psi(\mathbf{r}),\psi^\dagger(\mathbf{r}')]$ and appear to be highly untractable in their exact form. However they suggest, as a first approximation, a simplification which is precisely our model as follows:

## Algebra of the Model

We divide the volume $\Omega$ of the system into $M$ cubical cells of size $d^3$,

$$\Omega=Md^3, \tag{1.1}$$

where $d$ corresponds to the hard-core diameter (eventually we are of course interested in the limit $\Omega$, $M\rightarrow\infty$, with $d$ fixed), and define field amplitudes and occupation numbers for each cell:

$$n_j=\psi^\dagger{}_j\psi_j;\quad N_{\text{op}}=\textstyle\sum_j n_j,\quad (j=1,\cdots,M). \tag{1.2}$$

$N_{\text{op}}$ is the total particle number operator. We have, as usual for a discrete set of Bose operators,

$$[\psi_i,\psi_j]=0;\quad [\psi_i,n_j]=\psi_i\delta_{ij}, \tag{1.3}$$

but introduce the hard-core constraint approximately by limiting the eigenvalues of the $n_j$ to 0 and 1, i.e.,

$$\psi_j^2=0,\quad n_j^2=n_j. \tag{1.4}$$

This, together with (1.3), implies the basic commutation relation

$$[\psi_i,\psi_j^\dagger]=(1-2n_i)\delta_{i,j}. \tag{1.5}$$

The algebra of the model, as defined by (1.2)–(1.5), is equivalent to that of a set of Pauli operators: If we define "pseudospin" operators for each cell by

$$\sigma_j^{(1)}=\psi_j+\psi_j^\dagger,\quad \sigma_j^{(2)}=i(\psi_j^\dagger-\psi_j),\quad \sigma_j^{(3)}=1-2n_j, \tag{1.6}$$

the $\boldsymbol{\sigma}_j$ satisfy the standard relations for a set of independent Pauli spins:

$$\begin{aligned}&\sigma_j^{(\alpha)2}=1\quad (\alpha=1,2,3);\\ &\sigma_j^{(\alpha)}\sigma_j^{(\beta)}=i\sigma_j^{(\gamma)}\quad (\alpha,\beta,\gamma,\text{ cyclic});\\ &[\boldsymbol{\sigma}_i,\boldsymbol{\sigma}_j]=0\quad (i\neq j).\end{aligned} \tag{1.7}$$

[7] S. T. Beliaev, Zh. Eksperim. i Teor. Fiz. **34**, 417, 433 (1958) [translation: Soviet Phys.—JETP **7**, 289, 299 (1958)]; see also N. M. Hugenholtz and D. Pines, Phys. Rev. **116**, 489 (1959).

## Hamiltonian

The Hamiltonian to be used in the cell approximation is to some extent undetermined. The simplest choice, following Matsubara and Matsuda, is to replace the continuum kinetic energy operator by its finite-difference approximation

$$T=(\hbar^2/2m)\int_\Omega (\nabla\Psi^\dagger)\cdot(\nabla\Psi)d^3r\rightarrow(\hbar^2/2md^2)\times\textstyle\sum_{\langle ij\rangle}(\Psi_i^\dagger-\Psi_j^\dagger)(\Psi_i-\Psi_j),$$

where $\langle ij\rangle$ stands for nearest-neighbor pairs in the cubic lattice space (each cell has 6 nearest neighbors). Similarly, the potential energy becomes

$$\tfrac{1}{2}\textstyle\sum_{j\neq i} v_{ij}n_in_j.$$

The $v_{ij}$ involve only the attractive part of the interaction. For reasons of computational simplicity we make the further nonessential simplification of taking $v_{ij}$ to be zero for all except nearest-neighbor cells, where it takes the value $-v$. Thus, we have

$$H=(\hbar^2/md^2)\{3N_{\text{op}}-\tfrac{1}{2}\textstyle\sum_{\langle ij\rangle}(\psi_j^\dagger\psi_i+\psi_i^\dagger\psi_j)\}-v\textstyle\sum_{\langle ij\rangle}n_in_j, \tag{1.8}$$

and, of course,

$$[N_{\text{op}},H]=0. \tag{1.9}$$

To show explicity the effect of the hard-core constraint, it is convenient to express $H$ partly in terms of the pseudospin operators (1.6). One has, for $i\neq j$,

$$\begin{aligned}\tfrac{1}{2}(\psi_j^\dagger\psi_i+\psi_i^\dagger\psi_j)&=\tfrac{1}{4}[\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j-\sigma_i^{(3)}\sigma_j^{(3)}]\\&=\tfrac{1}{4}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j-1)+\tfrac{1}{2}(n_i+n_j)-n_in_j,\end{aligned}$$

so that

$$(md^2/\hbar^2)H=\textstyle\sum_{\langle ij\rangle}\tfrac{1}{4}(1-\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)+A\textstyle\sum_{\langle ij\rangle}n_in_j=H_0+AH_1, \tag{1.10}$$

with

$$A=1-v(md^2/\hbar^2). \tag{1.11}$$

In the following we shall express energies in units of $\hbar^2/md^2$.

In the cell approximation the drastic effects of the hard-core interaction thus appear in a simple intuitive form. In particular, there exists, in the hard-core case, a group of simple canonical transformations—the pseudospin rotations—which mix field amplitudes ($\sigma_j^{(1)}$ and $\sigma_j^{(2)}$) and particle densities ($\sigma_j^{(3)}$). This has consequences which are basic to the interest of the model:

(1) The part $H_0$ of the Hamiltonian—corresponding to the isotropic Heisenberg ferromagnet—being invariant under uniform rotation of the pseudospins, one can obtain a degenerate many-particle ground state of $H_0$ for any mean density $\langle N\rangle/\Omega\leq d^{-3}$ by such a rotation, Eqs. (2.23)–(2.25), from the vacuum (which corresponds to all the pseudospins "up"). This "unperturbed quasiparticle vacuum" takes the place of the Bose-condensed free-particle ground state as the starting

point of a perturbation treatment, the long-range order appearing as pseudospin alignment. There is no need to *assume* the persistence of Bose-Einstein condensation in the presence of the hard-core interaction at finite densities, though we shall see (Sec. 2) that something like a Bose-Einstein condensation does, in fact, persist. Note that the "unperturbed" system, described by $H_0$, includes the hard-core interaction plus a "square-well" attraction of depth $\hbar^2/md^2$ between particles in adjacent cells. As has been shown by Dyson[8] for spin waves[9] in the isotropic Heisenberg ferromagnet, this corresponds to zero scattering length for long-wavelength particles. Because of the "isotropy" of $H_0$ it also corresponds to zero scattering length for long-wavelength quasiparticles at any density. The perturbation parameter $A$, Eq. (1.11), measures the deviation of the attractive interaction from this unperturbed value. As in the treatments starting from a Bose-condensed free-particle state,[4–7] the present treatment starting from a homogeneous ground state of $H_0$ is valid only for positive scattering length,[10] i.e., for $A\geq 0$.

Since both $H_0$ and $AH_1$ (for $A\geq 0$) are positive semidefinite operators this means that, with the simple form of the Hamiltonian we are using, the pseudospin model cannot describe a "liquid" many-body bound state. This is not, however, an inherent limitation of the model, but could be remedied by including a longer range part in the attraction $v_{ij}$.

(2) Because of the mixing of amplitudes and densities by the pseudospin rotation from the vacuum, the quasiparticle excitations at finite density—corresponding to spin-wave excitations from a "rotated" ferromagnetic ground state—explicitly involve collective density fluctuations. Thus, because of the pseudospin algebra imposed by the hard-core constraint, the well-known density fluctuation character of the elementary excitations in the interacting Bose system emerges in the present model as the consequence of a simple canonical transformation. In the unperturbed system the density fluctuations have a free particle-like energy spectrum, since the ground state has infinite compressibility (its energy is density-independent), just as in the ideal Bose gas. The characteristic phonon spectrum emerges in the presence of the perturbing Hamiltonian.

The many-body ground state and singly excited states of the isotropic Hamiltonian $H_0$ are treated in Sec. 2.

It might appear from the preceding discussion that the presence of a hard core in the interaction may actually provide a conceptual simplification of the many-body Bose problem. This may be true, but only in part. The advantage of being able to start from an "unperturbed" ground state and "elementary excitations," which are much closer to those of the complete Hamiltonian than the usual free-particle states, is paid for in the present model with the formidable complications which the pseudospin algebra imposes on the formulation of a systematic perturbation procedure. These complications are well known in the theory of ferromagnetic spin waves. A detailed discussion for the isotropic case is given by Dyson.[8]

Because of these complications we confine our treatment of the full Hamiltonian, in Sec. 3, to an evaluation of the ground-state energy and of the elementary excitation spectrum by the equation-of-motion method in the random-phase approximation,[11] which is equivalent to the lowest order decoupling approximation in a Green's function treatment.[12] We find that this approximation gives an expansion for the ground-state energy in terms of the perturbation parameter $A$, which is self-consistent to order $A^{5/2}$ [Eq. (3.32)]. In the low-density limit this expansion, when expressed in terms of the scattering length $f_0$ [Eq. (3.34)], agrees with the well-known hard-sphere result of Lee, Huang, and Yang.[4] However, there is a higher density correction which arises from the fact that the *unperturbed* Hamiltonian, even though it has zero scattering length and zero $N$-particle ground-state *energy*, nevertheless has a ground-state *wave function* which at finite densities differs essentially from the completely Bose-condensed free-particle ground state (see Sec. 2).

### Quasimomentum Representation

We impose periodic boundary conditions and define the Fourier transforms of the local field operators in the usual way by

$$b_{\mathbf{k}}^{\dagger}=M^{-1/2}\sum_j \psi_j^{\dagger}\exp[i\mathbf{k}\cdot\mathbf{r}_j],\qquad b_{\mathbf{k}}=M^{-1/2}\sum_j \psi_j\exp[-i\mathbf{k}\cdot\mathbf{r}_j],\tag{1.12}$$

where the $\mathbf{r}_j$ are the lattice vectors of the cell centers, and the $\mathbf{k}$ range over the first zone of the reciprocal lattice. Similarly,

$$\rho_{\mathbf{k}}=M^{-1}\sum_j n_j\exp[-i\mathbf{k}\cdot\mathbf{r}_j];\quad \rho_{-\mathbf{k}}=\rho_{\mathbf{k}}^{\dagger}.\tag{1.13}$$

Note the normalization in (1.13), which is chosen to make $\rho_0$ the mean cell occupation number. Because of the hard-core constraint (1.4) the amplitudes for different $\mathbf{k}$ are not independent. One has

$$\sum_{\mathbf{k}} b_{\mathbf{k}}b_{\mathbf{l}-\mathbf{k}}=0;\quad \sum_{\mathbf{k}}\rho_{\mathbf{k}}\rho_{\mathbf{l}-\mathbf{k}}=\rho_{\mathbf{l}}\ (\text{all }\mathbf{l}).\tag{1.14}$$

[8] F. J. Dyson, Phys. Rev. **102**, 1217, 1230 (1956).

[9] F. Bloch, Z. Physik **61**, 206 (1930); T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).

[10] For $A<0$, in the limit $\Omega$, $N\rightarrow\infty$ where one can neglect surface effects, one easily sees that the true many-particle ground state of the Hamiltonian (1.10) is inhomogeneous, consisting of a "crystal" of maximum density ($\rho d^3=1$) and volume $Nd^3$, with binding energy per particle $3A$, the rest of the volume being empty.

[11] See, e.g., D. Pines, *The Many-Body Problem* (W. A. Benjamin, Inc., New York, 1961).

[12] N. N. Bogoliubov and S. V. Tyablikov, Dokl. Akad. Nauk SSSR **126**, 53 (1959) [translation: Soviet Phys.—Dokl. **4**, 589 (1959)]; R. A. Tahir-Kheli and D. Ter Haar, Phys. Rev. **127**, 88, 95 (1962).

The commutation relations are

$$[b_{\mathbf{k}},b_{\mathbf{l}}]=[b_{\mathbf{k}}^{\dagger},b_{\mathbf{l}}^{\dagger}]=0\,,\qquad [b_{\mathbf{k}},\rho_{\mathbf{l}}]=M^{-1}b_{\mathbf{k}+\mathbf{l}}\,, \tag{1.15}$$

and

$$[b_{\mathbf{k}},b_{\mathbf{l}}^{\dagger}]=\delta_{\mathbf{k},\mathbf{l}}-2\rho_{\mathbf{k}-\mathbf{l}}\,. \tag{1.16}$$

Equations (1.15) are as usual for Bose operators, but (1.14) and (1.16) reflect the hard-core constraint.

Using (1.14) for $\mathbf{l}=0$, we can write the Hamiltonian in the form

$$H_0=\sum_{\mathbf{k}}\{\tfrac{1}{2}(\gamma_0-\gamma_{\mathbf{k}})b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}-\tfrac{1}{2}M\gamma_{\mathbf{k}}\rho_{\mathbf{k}}^{\dagger}\rho_{\mathbf{k}}\}\,, \tag{1.17}$$
$$AH_1=\tfrac{1}{2}MA\sum_{\mathbf{k}}\gamma_{\mathbf{k}}\rho_{\mathbf{k}}^{\dagger}\rho_{\mathbf{k}}\,,$$

where

$$\gamma_{\mathbf{k}}=\sum_{\boldsymbol{\delta}}\exp(i\mathbf{k}\cdot\boldsymbol{\delta})\,; \tag{1.18}$$

$\boldsymbol{\delta}$ is a nearest-neighbor lattice vector, and the sum over $\boldsymbol{\delta}$ is over the six nearest neighbors of a given lattice cell. Note that $\gamma_0=6$ and

$$\sum_{\mathbf{k}}\gamma_{\mathbf{k}}=0\,. \tag{1.18$'$}$$

The property (1.18′) of the coefficients of the $\rho_{\mathbf{k}}^{\dagger}\rho_{\mathbf{k}}$ terms in the Hamiltonian is important. It reflects the fact that $v_{ii}=0$ and would continue to hold if we relaxed the limitation to nearest-neighbor attraction.

To avoid misunderstanding it should be stressed again that, because of the pseudospin algebra of the field operators, the "unperturbed" Hamiltonian $H_0$ *must* contain "interaction" terms ($\rho_{\mathbf{k}}^{\dagger}\rho_{\mathbf{k}}$) as well as "kinetic-energy" terms ($b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}$). For the same reason, the interpretation of the $b_{\mathbf{k}}^{\dagger},b_{\mathbf{k}}$ as creation and destruction operators of particles with quasimomentum $\mathbf{k}$ must be handled with caution. It is true that

$$N_{\mathrm{op}}=M\rho_0=\sum_{\mathbf{k}}\nu_{\mathbf{k}}\,;\quad \nu_{\mathbf{k}}=b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\,, \tag{1.19}$$

and that

$$[\nu_{\mathbf{k}},N_{\mathrm{op}}]=0\,, \tag{1.20}$$

$$[b_{\mathbf{k}},N_{\mathrm{op}}]=M[b_{\mathbf{k}},\rho_0]=b_{\mathbf{k}}\,. \tag{1.21}$$

Thus, the $b_{\mathbf{k}}$ ($b_{\mathbf{k}}^{\dagger}$) do indeed destroy (create) a particle. However, because of (1.16), neither $[b_{\mathbf{k}},\nu_{\mathbf{l}}]$ nor $[\nu_{\mathbf{k}},\nu_{\mathbf{l}}]$ is zero for $\mathbf{k}\neq\mathbf{l}$. Thus, except for $N=0$ or 1, an eigenstate of $N_{\mathrm{op}}$ can be at most an eigenstate of only one $\nu_{\mathbf{k}}$. Because of the incorporation of the hard-core interaction into the *kinematics* there are no "free" many-particle states in this mode. Nevertheless, the "occupation number" $\nu_0$ of the zero quasimomentum level turns out to be an important parameter for describing the ground-state properties of the system.

It should be pointed out that Matsubara and Matsuda,[1] who were interested mainly in the statistical mechanics of the λ transition, arrived at the pseudospin model from a starting point different from ours. They generalized the classical "lattice-gas" model, familiar in statistical mechanics, to include the effects of the quantum-mechanical zero-point motion. The relation of the classical lattice-gas model to the pseudospin model is, in fact, the same as that of the Ising model to the Heisenberg ferromagnet.

Finally, we note that a pseudospin representation which in some respects resembles the present one has been used by Anderson and by Wada, Takano, and Fukuda[13] to treat the BCS theory of superconductivity. There the pseudospin operators represent the bound Cooper electron pairs. The constraint which arises from the hard-core interaction for the Bose system comes from the exclusion principle for the electron-pair system. However, there are important differences between the two pseudospin models. The Anderson pseudospins are defined in *momentum space*, and the BCS interaction between them in momentum space is long range, so that the elementary excitations in a "molecular field" approximation correspond to "local" (in momentum space) spin flips, giving the energy gap for quasiparticle excitations in superconductors. In the hard-core Bose model the pseudospins are local in coordinate space and the interaction between them is *short range*, so that the elementary excitations correspond to collective nonlocal spin deviations, i.e., "pseudospin waves," with no energy gap. It seems interesting that the frequently observed similarities between superconductivity and superfluidity,[14] together with their characteristic differences, should here crop up again in a new context.

## 2. UNPERTURBED HAMILTONIAN; *N*-PARTICLE GROUND STATE AND ELEMENTARY EXCITATIONS

We begin by determining the exact ground state $\Phi_0(N)$ of the unperturbed system, described by $H_0$, for a given number of particles. It is convenient to consider $H_0$ in the pseudospin form,

$$H_0=\tfrac{1}{4}\sum_{\langle ij\rangle}(1-\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)\,, \tag{1.10$'$}$$

and to define the total pseudospin operator

$$\mathbf{S}=\tfrac{1}{2}\sum_j\boldsymbol{\sigma}_j\,, \tag{2.1}$$

which, from (1.6) and (1.12), has components

$$S^{(1)}=\tfrac{1}{2}M^{1/2}(b_0^{\dagger}+b_0)\,;\quad S^{(2)}=\tfrac{1}{2}iM^{1/2}(b_0^{\dagger}-b_0)\,;\quad S^{(3)}=\tfrac{1}{2}M-N_{\mathrm{op}}\,. \tag{2.2}$$

Note that

$$\mathbf{S}^2=M\nu_0+S^{(3)}(S^{(3)}+1) \tag{2.3}$$

with $\nu_{\mathbf{k}}$ defined in (1.19). Since

$$[H_0,\mathbf{S}]=0\,, \tag{2.4}$$

we can consider simultaneous eigenstates of $H_0,S^{(3)}$

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[13] P. W. Anderson, Phys. Rev. **112**, 1900 (1958); Y. Wada, F. Takano, and N. Fukuda, Progr. Theoret. Phys. (Kyoto) **19**, 597 (1958). See also K. Baumann, G. Eder, R. Sexl, and W. Thirring, Ann. Phys. (N. Y.) **16**, 14 (1961).

[14] See F. London, *Superfluids* (John Wiley & Sons, Inc., New York, 1950), Vol. I; (1954), Vol. II.

(i.e., $N_{op}$), and $\mathbf{S}^2$. Because of (2.3) these will be eigenstates of $\nu_0$ as well.

In terms of the spin picture the nature of $\Phi_0(N)$ is determined easily. Because of the isotropy of $H_0$, its eigenvalues are independent of the number of particles; the lowest eigenvalue, zero, belongs to those states which are symmetric in all nearest-neighbor spin pairs. But, since the interchange of *any* two spins can be accomplished by a sequence of nearest-neighbor spin interchanges, $\Phi_0(N)$ must be symmetric in *all* spins, that is, $S=\frac{1}{2}M$. The physical vacuum itself is the "ferromagnetic" ground state in which all $M$ spins are "up" so that

$$\Phi_0(0)=|0\rangle=|S=\tfrac{1}{2}M;\quad S^{(3)}=\tfrac{1}{2}M\rangle. \tag{2.5}$$

$\Phi_0(N)$ is obtained from (2.5) by $N$-fold application of the $S^{(3)}$-lowering operator $b_0^\dagger$. The unperturbed $N$-particle ground state is, then,

$$\Phi_0(N)=|S=\tfrac{1}{2}M;\; S^{(3)}=\tfrac{1}{2}M-N\rangle=D_0^{(N)}(b_0^\dagger)^N|0\rangle, \tag{2.6}$$

or, in terms of the local operators,

$$\Phi_0(N)=M^{-N/2}D_0^{(N)}(\textstyle\sum_j\psi_j^\dagger)^N|0\rangle, \tag{2.6$'$}$$

where the normalization is

$$D_0^{(N)}=M^{N/2}[N!(_N{}^M)^{1/2}]^{-1}. \tag{2.7}$$

$\Phi_0(N)$ satisfies

$$\begin{aligned}&H_0\Phi_0(N)=0,\\&S^2\Phi_0(N)=\tfrac{1}{2}M(\tfrac{1}{2}M+1)\Phi_0(N).\end{aligned} \tag{2.8}$$

The physical nature of $\Phi_0(N)$ is easily seen from the form (2.6′). Because of the hard-core constraint, Eq. (1.4), $\Phi_0(N)$ is a symmetric sum of $(_N{}^M)$ product states in each of which $N$ cells are singly occupied (spins down) and $M$-$N$ cells are vacant (spins up). Denoting expectation values in $\Phi_0(N)$ by $\langle\ \rangle_0$, the mean occupation number for any cell is

$$\langle n_j\rangle_0=\rho_0\equiv\rho d^3,\quad \text{all } j, \tag{2.9}$$

with $\rho=N/\Omega$, the mean number density. There is no density correlation between cells, so that

$$\langle n_l n_m\rangle_0\approx(\rho d^3)^2,\quad l\neq m. \tag{2.10}$$

The expectation value of the perturbing Hamiltonian is

$$W_1\equiv A\langle H_1\rangle_0=A\textstyle\sum_{\langle ij\rangle}\langle n_in_j\rangle_0=\tfrac{1}{2}\gamma_0AN\rho d^3. \tag{2.11}$$

### "Bose-Einstein Condensation"

From (2.3), $\Phi_0(N)$ is also an eigenstate of the "number of particles with zero quasimomentum":

$$\begin{aligned}\nu_0\Phi_0(N)&=N[1-(N-1)/M]\Phi_0(N)\\&\approx N(1-\rho d^3)\Phi_0(N).\end{aligned} \tag{2.12}$$

To calculate the mean values of the other $\nu_{\mathbf{k}}$, we first express $\nu_{\mathbf{k}}$ in terms of the local operators

$$\nu_{\mathbf{k}}=N_{op}/M+M^{-1}\textstyle\sum_{i\neq j}\psi_j^\dagger\psi_i\exp[i\mathbf{k}\cdot(\mathbf{r}_j-\mathbf{r}_i)].$$

From the nature of $\Phi_0(N)$ one sees that $\langle\psi_j^\dagger\psi_i\rangle_{0(i\neq j)}$ is independent of $i$ and $j$. Thus,

$$\begin{aligned}\langle\nu_{\mathbf{k}}\rangle_0&=\rho d^3-\langle\psi_j^\dagger\psi_i\rangle_{0(i\neq j)}\\&=\rho d^3-\tfrac{1}{4}\{\langle\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\rangle_0-\langle(1-2n_i)(1-2n_j)\rangle_0\}_{i\neq j}\\&\approx(\rho d^3)^2,\quad \mathbf{k}\neq0.\end{aligned} \tag{2.13}$$

In the limit $N,M\to\infty$ we have

$$\begin{aligned}&\langle\nu_0\rangle_0/N\to1-\rho d^3,\\&\langle\nu_{\mathbf{k}}\rangle_0/N\to0,\quad \mathbf{k}\neq0.\end{aligned}$$

The depletion $\xi_0$ of the free-particle ground state in the ground state of $H_0$ is

$$\xi_0\equiv\textstyle\sum_{k\neq0}\langle\nu_{\mathbf{k}}\rangle_0/N\to\rho d^3. \tag{2.14}$$

Thus, for all $\rho d^3<1$, the unperturbed ground state exhibits "Bose-Einstein condensation" in quasimomentum space. The depletion is clearly an "excluded-volume" effect of the hard-core constraint (1.4).

### Elementary Excitations

We now construct eigenstates of $H_0$ containing a single elementary excitation. In the spin picture an excitation is a "pseudospin wave" in which $S$ is lowered by one unit from its ground-state value $\frac{1}{2}M$, while $S^{(3)}$ is maintained at $\frac{1}{2}M-N$. Consider the normalized states ($\mathbf{k}\neq0$)

$$\Phi_{\mathbf{k}}(N)=D_{\mathbf{k}}^{(N)}\rho_{\mathbf{k}}^\dagger(b_0^\dagger)^N|0\rangle\sim\rho_{\mathbf{k}}^\dagger\Phi_0(N), \tag{2.15}$$

where the density fluctuation operator $\rho_{\mathbf{k}}^\dagger$ is defined by (1.13) and $D_{\mathbf{k}}^{(N)}$ is the normalization which we suppress below for simplicity. These states are orthogonal to the ground state and to each other,

$$\begin{aligned}&\langle\Phi_0(N)|\Phi_{\mathbf{k}}(N)\rangle=0,\quad \mathbf{k}\neq0,\\&\langle\Phi_{\mathbf{k}}(N)|\Phi_{\mathbf{l}}(N)\rangle=\delta_{\mathbf{k},\mathbf{l}},\quad \mathbf{k}\neq0.\end{aligned} \tag{2.16}$$

Since $\rho_{\mathbf{k}}^\dagger$ commutes with $S^{(3)}$, the states (2.15) are eigenstates of $S^{(3)}$ with eigenvalue $\frac{1}{2}M-N$. To show that these states are also eigenstates of $H_0$, we write first,

$$\rho_{\mathbf{k}}^\dagger(b_0^\dagger)^N|0\rangle=(N/M)b_{\mathbf{k}}^\dagger(b_0^\dagger)^{N-1}|0\rangle, \tag{2.17}$$

which results by using the commutator (1.15) a total of $N$ times so as to commute $\rho_{\mathbf{k}}^\dagger$ through $(b_0^\dagger)^N$ to act on the vacuum. Because $H_0$ and $b_{\mathbf{k}}^\dagger$ commute with $b_0^\dagger$, $H_0$ operating on (2.17) yields

$$H_0\rho_{\mathbf{k}}^\dagger(b_0^\dagger)^N|0\rangle=(N/M)(b_0^\dagger)^{N-1}[H_0,b_{\mathbf{k}}^\dagger]|0\rangle.$$

Expressing $H_0$ in the form (1.17), one finds for this last commutator

$$[H_0,b_{\mathbf{k}}^\dagger]=\epsilon_{\mathbf{k}}{}^0b_{\mathbf{k}}^\dagger+2\textstyle\sum_{\mathbf{l}}(\epsilon_{\mathbf{k}-\mathbf{l}}{}^0-\epsilon_{\mathbf{l}}{}^0)b_{\mathbf{l}}^\dagger\rho_{\mathbf{k}-\mathbf{l}}^\dagger,$$

with

$$\epsilon_{\mathbf{k}}{}^0=\tfrac{1}{2}(\gamma_0-\gamma_{\mathbf{k}}). \tag{2.18}$$

Combining (2.15), (2.17), and (2.18), we obtain

$$H_0\Phi_{\mathbf{k}}(N)=\epsilon_{\mathbf{k}}{}^0\Phi_{\mathbf{k}}(N). \quad (2.19)$$

In general, the energy $\epsilon_{\mathbf{k}}{}^0$ of the excitation generated by $\rho_{\mathbf{k}}{}^\dagger$ is anisotropic in $\mathbf{k}$ due to the effect of the "lattice structure" but, for wavelengths long compared to the lattice spacing $d$, the energy spectrum is isotropic and free-particle-like, reflecting the fact that the ground state has infinite compressibility:

$$\epsilon_{\mathbf{k}}{}^0\approx\tfrac{1}{2}k^2d^2, \quad kd\ll 1. \quad (2.20)$$

(In ordinary units $\epsilon_{\mathbf{k}}{}^0\approx\hbar^2k^2/2m$.) In $\Phi_{\mathbf{k}}(N)$ one particle has been excited from the condensate;

$$\nu_0\Phi_{\mathbf{k}}(N)=[N(1-\rho d^3)-1]\Phi_{\mathbf{k}}(N), \quad \mathbf{k}\neq 0. \quad (2.21)$$

Thus, using (2.3),

$$\mathbf{S}^2\Phi_{\mathbf{k}}(N)=\tfrac{1}{2}M(\tfrac{1}{2}M-1)\Phi_{\mathbf{k}}(N), \quad k\neq 0, \quad (2.22)$$

so that in our previous notation

$$\Phi_{\mathbf{k}}(N)=|S=\tfrac{1}{2}M-1;\ S^{(3)}=\tfrac{1}{2}M-N\rangle. \quad (2.15')$$

As is well known in spin-wave theory, states containing more than one spin-wave excitation are not orthogonal to each other (kinetmatical interaction) and $H_0$ is not diagonal in such states, i.e., the excitations scatter one another (dynamical interaction). For long wavelengths this scattering is very weak and $H_0$ is approximately diagonal in such states. A complete discussion is given by Dyson[8] for isotropic Heisenberg exchange. With anisotropy present the scattering is not so weak, the two-body scattering length at zero energy being finite (see Appendix II).

### Unperturbed Quasiparticle Vacuum

For the purpose of treating the full Hamiltonian (Sec. 3) it is convenient, as usual, to relax the restriction to eigenstates of $N$ (i.e., of $S^{(3)}$). As was explained in the Introduction, this allows, in the present model, the use of particularly simple many-body eigenstates of $H_0$: Consider a uniform rotation of the pseudospins about the 2-axis through an angle $\theta$.

$$R_2(\theta)=\exp[-i\theta S^{(2)}]=\textstyle\prod_j(\cos\tfrac{1}{2}\theta-i\sigma_j{}^{(2)}\sin\tfrac{1}{2}\theta). \quad (2.23)$$

The state obtained from the vacuum by this transformation is

$$|\theta\rangle=R_2(\theta)|0\rangle=\textstyle\prod_j(\cos\tfrac{1}{2}\theta+\psi_j{}^\dagger\sin\tfrac{1}{2}\theta)|0\rangle. \quad (2.24)$$

Denoting axes in the rotated coordinate systems by $x$, $y$, $z$, the relations between the original and transformed Pauli operators are

$$\begin{aligned}\sigma_j{}^{(1)}&=(\cos\theta)\sigma_j{}^x-(\sin\theta)\sigma_j{}^z;\\ \sigma_j{}^{(2)}&=\sigma_j{}^y;\\ \sigma_j{}^{(3)}&=(\cos\theta)\sigma_j{}^z+(\sin\theta)\sigma_j{}^x.\end{aligned} \quad (2.25)$$

In $|\theta\rangle$ all the pseudospins are aligned along the $z$ direction;

$$\sigma_j{}^z|\theta\rangle=(R_2\sigma_j{}^{(3)}R_2{}^{-1})R_2|0\rangle=|\theta\rangle, \quad (2.26)$$

and, since $H_0$ and $\mathbf{S}^2$ are invariant under uniform rotation,

$$\begin{aligned}&H_0|\theta\rangle=0,\\ &\mathbf{S}^2|\theta\rangle=\tfrac{1}{2}M(\tfrac{1}{2}M+1)|\theta\rangle.\end{aligned} \quad (2.27)$$

However, (2.24) is not an eigenstate of $S^{(3)}$ and so of the number of particles. (The zero-temperature chemical potential of the unperturbed system is obviously zero and so need not be introduced.) The mean value of $S^{(3)}$ is

$$\langle S^{(3)}\rangle_\theta=\tfrac{1}{2}M\cos\theta. \quad (2.28)$$

Then, the mean number of particles will be described correctly if

$$\cos\theta=1-2\rho_0. \quad (2.29)$$

A simple relationship exists between $\Phi_0(N)$ and $|\theta\rangle$, which is degenerate with all those states $|\theta;\phi\rangle$ which can be generated by rotating $|\theta\rangle$ about the 3-direction through azimuthal angles $\phi, 0<\phi<2\pi$. This "cone degeneracy" persists in the *full* Hamiltonian which, though not isotropic, still commutes with $N_{\text{op}}$, and thus with $S^{(3)}$. As is well known (cf., e.g., Thirring *et al.*[13]), $\Phi_0(N)$ can be written as a superposition of the states $|\theta;\phi\rangle$. The state $|\theta\rangle$, itself, is a superposition of ground states corresponding to different eigenvalues of $N_{\text{op}}$ as can be seen by expanding the product in (2.24):

$$|\theta\rangle=(\cos\tfrac{1}{2}\theta)^M\sum_{l=0}^{M}\frac{(\tan\frac{1}{2}\theta)^l}{l!}(\textstyle\sum_j\Psi_j{}^\dagger)^l|0\rangle. \quad (2.24')$$

Thus, except for normalization, $\Phi_0(N)$ is the projection of $|\theta\rangle$ onto the subspace $N_{\text{op}}=N$:

$$|\theta\rangle_N\sim\Phi_0(N). \quad (2.30)$$

In the limit $M,N\rightarrow\infty$ the distribution of $N$ in (2.24′) is sharply peaked at the mean value. One has

$$(\Delta N)_\theta{}^2=\langle N_{\text{op}}{}^2\rangle_\theta-\langle N_{\text{op}}\rangle_\theta{}^2=\langle S^{(3)2}\rangle_\theta-\langle S^{(3)}\rangle_\theta{}^2.$$

Using (2.29), a simple calculation gives in the limit

$$(\Delta N)^2\approx\tfrac{1}{4}M\sin^2\theta=N(1-\rho d^3),$$

so the mean fluctuation,

$$\Delta N/N=[(1-\rho d^3)/N]^{1/2}, \quad (2.31)$$

vanishes as $N\rightarrow\infty$ and $|\theta\rangle$ is a good unperturbed many-particle ground state for the infinite system.

The chief virtue of the uniform rotation (2.23) lies in the fact that, for any density, $|\theta\rangle$ is the vacuum for the transformed creation, destruction operators

$$\begin{aligned}\phi_j{}^\dagger&=R_2(\theta)\psi_j{}^\dagger R_2{}^{-1}(\theta)=\tfrac{1}{2}(\sigma_j{}^x-i\sigma_j{}^y),\\ \phi_j&=\tfrac{1}{2}(\sigma_j{}^x+i\sigma_j{}^y),\end{aligned} \quad (2.32)$$

with Fourier transforms

$$C_{\mathbf{k}}{}^{\dagger}=R_2(\theta)b_{\mathbf{k}}{}^{\dagger}R_2{}^{-1}(\theta)=M^{-1/2}\sum_j \phi_j{}^{\dagger}\exp[i\mathbf{k}\cdot\mathbf{r}_j],\quad (2.32')$$

$$C_{\mathbf{k}}=M^{-1/2}\sum_j \phi_j \exp[-i\mathbf{k}\cdot\mathbf{r}_j].$$

For, from the unitary property of the transformation,

$$\phi_j|\theta\rangle=0,\quad \text{all } j;\qquad C_{\mathbf{k}}|\theta\rangle=0,\quad \text{all } \mathbf{k}. \quad (2.33)$$

Since $H_0$ is invariant under the transformation and the commutation relations between the $C_{\mathbf{k}}$, $C_{\mathbf{k}}{}^{\dagger}$ have the same form as those between the $b_{\mathbf{k}}$, $b_{\mathbf{k}}{}^{\dagger}$, the state

$$C_{\mathbf{k}}{}^{\dagger}|\theta\rangle\equiv|\theta;\mathbf{k}\rangle \quad (2.34)$$

containing one quasiparticle created by $C_{\mathbf{k}}{}^{\dagger}$ is an eigenstate of $H_0$:

$$H_0|\theta;\mathbf{k}\rangle=\epsilon_{\mathbf{k}}{}^0|\theta;\mathbf{k}\rangle. \quad (2.35)$$

Thus, the $C_{\mathbf{k}}{}^{\dagger}$ acting on $|\theta\rangle$ create "unperturbed quasiparticles" having the same energy spectrum as the density fluctuations $\rho_{\mathbf{k}}{}^{\dagger}$ acting on $\Phi_0(N)$. These quasiparticles are linear combinations of particles, holes, and density fluctuations, as can be seen by using (2.25) in (2.32), (2.32'):

$$\phi_j{}^{\dagger}=\tfrac{1}{2}(1+\cos\theta)\psi_j{}^{\dagger}-\tfrac{1}{2}(1-\cos\theta)\psi_j+\tfrac{1}{2}\sin\theta(1-2\psi_j{}^{\dagger}\psi_j),$$

or

$$C_{\mathbf{k}}{}^{\dagger}=\tfrac{1}{2}(1+\cos\theta)b_{\mathbf{k}}{}^{\dagger}-\tfrac{1}{2}(1-\cos\theta)b_{-\mathbf{k}}+\tfrac{1}{2}M^{1/2}\sin\theta(\delta_{\mathbf{k},0}-2\rho_{\mathbf{k}}{}^{\dagger}). \quad (2.36)$$

## 3. FULL HAMILTONIAN; RANDOM-PHASE APPROXIMATION

To relax the restriction to eigenstates of $N_{\text{op}}$ we introduce the chemical potential $\mu$ in the usual way, replacing $H$, Eq. (1.10), by

$$H'\equiv H-\mu N_{\text{op}}=H_0+AH_1-\mu N_{\text{op}}. \quad (3.1)$$

Remembering the cyclindrical symmetry of $H'$ about the 3-direction, we look for a homogeneous ground state $\Psi_0(A,\mu)$ of $H'$, analogous to the unperturbed quasiparticle vacuum $|\theta\rangle$, having a finite net "magnetization" $\sigma$ per cell in *some* direction $z$ in the 1–3 plane. Accordingly, we work with the rotated spin operators (2.25), in terms of which $H'$ is given by

$$H_0=\tfrac{1}{4}\sum_{\langle ij\rangle}(1-\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j), \quad (3.2a)$$

$$H_1=\tfrac{1}{4}\sum_{\langle ij\rangle}\{1-(\sigma_i{}^x+\sigma_j{}^x)\sin\theta-(\sigma_i{}^z+\sigma_j{}^z)\cos\theta+\sigma_i{}^x\sigma_j{}^x\sin^2\theta+\sigma_i{}^z\sigma_j{}^z\cos^2\theta+(\sigma_i{}^x\sigma_j{}^z+\sigma_j{}^x\sigma_i{}^z)\cos\theta\sin\theta\}, \quad (3.2b)$$

$$N_{\text{op}}=\tfrac{1}{2}\sum_i(1-\sigma_i{}^x\sin\theta-\sigma_i{}^z\cos\theta). \quad (3.2c)$$

We have

$$\langle\sigma_j{}^z\rangle=1-2M^{-1}\sum_{\mathbf{k}}\langle C_{\mathbf{k}}{}^{\dagger}C_{\mathbf{k}}\rangle\equiv\sigma, \quad (3.3a)$$

$$\langle\sigma_j{}^x\rangle=\langle\sigma_j{}^y\rangle=0, \quad (3.3b)$$

where $\langle\ \rangle$ denotes expectation values in $\Psi_0(A,\mu)$ and, therefore,

$$\rho_0=N/M=\tfrac{1}{2}(1-\sigma\cos\theta). \quad (3.4)$$

The parameters $\theta$ and $\sigma$ are to be determined self-consistently.

### Exact Ground-State Properties

The Heisenberg equations of motion for the pseudospin operators are

$$\hbar\dot{\sigma}_j{}^x=\tfrac{1}{2}(\sigma_j{}^y\sum_\delta\sigma_{j+\delta}{}^z-\sigma_j{}^z\sum_\delta\sigma_{j+\delta}{}^y)+\tfrac{1}{2}A\sigma_j{}^y\sum_\delta(\cos\theta-\sigma_{j+\delta}{}^z\cos^2\theta-\sigma_{j+\delta}{}^x\sin\theta\cos\theta)-\mu\sigma_j{}^y\cos\theta; \quad (3.5a)$$

$$\hbar\dot{\sigma}_j{}^y=\tfrac{1}{2}(\sigma_j{}^z\sum_\delta\sigma_{j+\delta}{}^x-\sigma_j{}^x\sum_\delta\sigma_{j+\delta}{}^z)+\tfrac{1}{2}A\{\sigma_j{}^z\sum_\delta(\sin\theta-\sigma_{j+\delta}{}^x\sin^2\theta-\sigma_{j+\delta}{}^z\sin\theta\cos\theta)-\sigma_j{}^x\sum_\delta(\cos\theta-\sigma_{j+\delta}{}^z\cos^2\theta-\sigma_{j+\delta}{}^x\sin\theta\cos\theta)\}-\mu(\sigma_j{}^z\sin\theta-\sigma_j{}^x\cos\theta); \quad (3.5b)$$

$$\hbar\dot{\sigma}_j{}^z=\tfrac{1}{2}(\sigma_j{}^x\sum_\delta\sigma_{j+\delta}{}^y-\sigma_j{}^y\sum_\delta\sigma_{j+\delta}{}^x)-\tfrac{1}{2}A\sigma_j{}^y\sum_\delta(\sin\theta-\sigma_{j+\delta}{}^x\sin^2\theta-\sigma_{j+\delta}{}^z\sin\theta\cos\theta)-\mu\sigma_j{}^y\sin\theta; \quad (3.5c)$$

the sums over $\delta$ range over the 6 nearest neighbors of $j$. Since $\Psi_0$ is an eigenstate of $H'$, it follows that $\langle[F,H']\rangle=0$ for any operator $F$. In particular, $\langle d\boldsymbol{\sigma}_j/dt\rangle=0$. Let

$$\sigma_j{}^z=\sigma+\tau_j, \quad (3.6)$$

where $\langle\tau_j\rangle=0$ by definition. Taking the expectation value of (3.5b) and using (3.4), we find the exact expression for the chemical potential:

$$\mu=A\{\gamma_0\rho_0+(\cos\theta/2\sigma)[\sum_\delta\langle\sigma_j{}^x\sigma_{j+\delta}{}^x\rangle-\sum_\delta\langle\tau_j\tau_{j+\delta}\rangle]+(\cos2\theta/2\sigma\sin\theta)\sum_\delta\langle\tau_j\sigma_{j+\delta}{}^x\rangle\}. \quad (3.7)$$

With this relation for $\mu$, the expectation values of the other two equations of motion both yield

$$\cos\theta\sum_\delta\langle\sigma_j{}^y\tau_{j+\delta}\rangle+\sin\theta\sum_\delta\langle\sigma_j{}^y\sigma_{j+\delta}{}^x\rangle=\sum_\delta\langle\sigma_j{}^{(2)}\sigma_{j+\delta}{}^{(3)}\rangle=0.$$

From the chemical potential the ground-state energy $E_0$ of $H'+\mu N_{\text{op}}$ can be obtained in the usual manner by

$$E_0(A,N)=M\int_0^{\rho_0}\mu d\rho_0,\quad (\text{constant } A), \quad (3.8)$$

where the integration is performed at constant interaction strength $A$. Alternatively, the ground-state energy can be obtained from the Pauli-Feynman theorem[11]

$$E_0(A,N)=\int_0^A\langle H_1\rangle dA,\quad (\text{constant } N), \quad (3.9)$$

by integrating at constant $N$. Equation (3.9) holds because $E_0(0,N)$ vanishes and

$$\langle H_1\rangle=(\partial E_0'/\partial A)_\mu=(\partial E_0/\partial A)_N,$$

where $E_0'(A,\mu)$ is the eigenvalue of $H'$ in $\Psi_0$. Combining (3.3), (3.4), and (3.6), the exact expression for $\langle H_1\rangle$ is

$$\langle H_1\rangle=\tfrac{1}{2}\gamma_0N_0\rho_0+(M/8)\textstyle\sum_\delta\{\langle\sigma_j^x\sigma_{j+\delta}^x\rangle\sin^2\theta +\langle\tau_j\tau_{j+\delta}\rangle\cos^2\theta+2\langle\tau_j\sigma_{j+\delta}^x\rangle\sin\theta\cos\theta\}. \quad (3.10)$$

We shall test the self-consistency of the random-phase approximation by comparing the results it gives in (3.8) and (3.9).

### Linearized Equations of Motion

To evaluate the ground-state energy and excitation spectrum for the perturbed system, we adopt the random-phase approximation[11] and linearize the equations of motion (3.5) about the state $\Psi_0$ by using (3.3) and (3.4). In this approximation, equivalent to the spin-wave approximation for a ferromagnet with anisotropic interaction, we find

$$\hbar\dot\sigma_j^x\approx\tfrac{1}{2}\sigma(\gamma_0\sigma_j^y-\textstyle\sum_\delta\sigma_{j+\delta}^y)+\sigma_j^y\cos\theta(A\gamma_0\rho_0-\mu); \quad (3.11\text{a})$$

$$\hbar\dot\sigma_j^y\approx\sigma\sin\theta(A\gamma_0\rho_0-\mu)-\tfrac{1}{2}\sigma[\gamma_0\sigma_j^x-(1-A\sin^2\theta)\times\textstyle\sum_\delta\sigma_{j+\delta}^x]-\sigma_j^x\cos\theta(A\gamma_0\rho_0-\mu); \quad (3.11\text{b})$$

$$\hbar\dot\tau_j\approx-\sigma_j^y\sin\theta(A\gamma_0\rho_0-\mu). \quad (3.11\text{c})$$

The requirement that the expectation values of Eqs. (3.11) vanish now determines the zero-order chemical potential as

$$\mu_0=A\gamma_0\rho_0, \quad (3.12)$$

which is equivalent to the first-order perturbation result (2.11). From (3.12) we see that $\dot\tau_j\approx 0$, i.e., $\tau_j$ is a second-order quantity.

Upon taking Fourier transforms of (3.11) and using (3.12) and (2.32), (2.32′), we have

$$\hbar\dot C_{\mathbf{k}}^\dagger\approx i\sigma\{\epsilon_{\mathbf{k}}^0C_{\mathbf{k}}^\dagger+\tfrac{1}{2}J\gamma_{\mathbf{k}}(C_{\mathbf{k}}^\dagger+C_{-\mathbf{k}})\}; \quad (3.13\text{a})$$

$$\hbar\dot C_{\mathbf{k}}\approx-i\sigma\{\epsilon_{\mathbf{k}}^0C_{\mathbf{k}}+\tfrac{1}{2}J\gamma_{\mathbf{k}}(C_{\mathbf{k}}+C_{-\mathbf{k}}^\dagger)\}, \quad (3.13\text{b})$$

where

$$J\equiv\tfrac{1}{2}A\sin^2\theta. \quad (3.14)$$

We note that these linearized equations of motion could be obtained from a reduced Hamiltonian of the Bogoliubov form,[6]

$$H_{\text{red}}=\tfrac{1}{2}\gamma_0AN\rho_0+\textstyle\sum_{\mathbf{k}}\{\epsilon_{\mathbf{k}}^0C_{\mathbf{k}}^\dagger C_{\mathbf{k}} +\tfrac{1}{2}J\gamma_{\mathbf{k}}[C_{\mathbf{k}}^\dagger C_{\mathbf{k}}+\tfrac{1}{2}(C_{\mathbf{k}}C_{-\mathbf{k}}+C_{-\mathbf{k}}^\dagger C_{\mathbf{k}}^\dagger)]\}, \quad (3.15)$$

if the $C_{\mathbf{k}}$, $C_{\mathbf{k}}^\dagger$ satisfied the commutation relations

$$[C_{\mathbf{k}},C_{\mathbf{k}'}^\dagger]\approx\sigma\delta_{\mathbf{k},\mathbf{k}'}, \quad (3.16)$$

which are equivalent to approximating $[\sigma_j^x,\sigma_{j'}^y]$ by

$$[\sigma_j^x,\sigma_{j'}^y]\approx\langle[\sigma_j^x,\sigma_{j'}^y]\rangle=2i\sigma\delta_{j,j'}. \quad (3.16')$$

Also, note that

$$\textstyle\sum_{\mathbf{k}}\gamma_{\mathbf{k}}[C_{\mathbf{k}}^\dagger C_{\mathbf{k}}+\tfrac{1}{2}(C_{\mathbf{k}}C_{-\mathbf{k}}+C_{-\mathbf{k}}^\dagger C_{\mathbf{k}}^\dagger)] =\tfrac{1}{2}\sum_{j,\delta}\sigma_j^x\sigma_{j+\delta}^x. \quad (3.17)$$

Equations (3.13), or equivalently $H_{\text{red}}$, are brought into diagonal form by a linear transformation of the Bogoliubov type, which is, however, only approximately canonical since (3.16) is not exact:

$$C_{\mathbf{k}}=\alpha_{\mathbf{k}}\cosh(\tfrac{1}{2}\chi_{\mathbf{k}})+\alpha_{-\mathbf{k}}^\dagger\sinh(\tfrac{1}{2}\chi_{\mathbf{k}}),\quad C_{\mathbf{k}}^\dagger=\alpha_{\mathbf{k}}^\dagger\cosh(\tfrac{1}{2}\chi_{\mathbf{k}})+\alpha_{-\mathbf{k}}\sinh(\tfrac{1}{2}\chi_{\mathbf{k}}), \quad (3.18)$$

with $\chi_{\mathbf{k}}=\chi_{-\mathbf{k}}$, and

$$[\alpha_{\mathbf{k}},\alpha_{\mathbf{k}'}^\dagger]\approx\sigma\delta_{\mathbf{k},\mathbf{k}'}. \quad (3.19)$$

The diagonalization condition is

$$(\gamma_0-\gamma_{\mathbf{k}}+J\gamma_{\mathbf{k}})\sinh\chi_{\mathbf{k}}+J\gamma_{\mathbf{k}}\cosh\chi_{\mathbf{k}}=0, \quad (3.20)$$

which yields

$$\cosh\chi_{\mathbf{k}}=(\gamma_0-\gamma_{\mathbf{k}}+J\gamma_{\mathbf{k}})[(\gamma_0-\gamma_{\mathbf{k}}+J\gamma_{\mathbf{k}})^2-(J\gamma_{\mathbf{k}})^2]^{-1/2},\quad \sinh\chi_{\mathbf{k}}=-J\gamma_{\mathbf{k}}[(\gamma_0-\gamma_{\mathbf{k}}+J\gamma_{\mathbf{k}})^2-(J\gamma_{\mathbf{k}})^2]^{-1/2}. \quad (3.21)$$

The excitation energy for a "pseudospin wave" of wave vector $\mathbf{k}$ is given by

$$\epsilon_{\mathbf{k}}=\tfrac{1}{2}\sigma[(\gamma_0-\gamma_k+J\gamma_k)\cosh\chi_{\mathbf{k}}+J\gamma_{\mathbf{k}}\sinh\chi_{\mathbf{k}}] =\tfrac{1}{2}\sigma[(\gamma_0-\gamma_{\mathbf{k}}+J\gamma_{\mathbf{k}})^2-(J\gamma_{\mathbf{k}})^2]^{1/2}. \quad (3.22)$$

Note that in the random-phase approximation $\epsilon_{\mathbf{k}}$ scales with the magnetization. This agrees with the lowest order decoupling result in a Green's function treatment.[12] However, as we shall see, the approximation is self-consistent only to the order in which the deviation of $\sigma$ from 1 can be neglected in the expressions (3.7) for $\mu$ and (3.10) for $H_1$. For wavelengths long compared to the lattice spacing $d$,$\gamma_{\mathbf{k}}$ can be replaced by the isotropic expression $\gamma_{\mathbf{k}}\approx\gamma_0-(kd)^2$; the energy spectrum then is linear in $k$ as is characteristic of phonon excitations:

$$\epsilon_{\mathbf{k}}\rightarrow\tfrac{1}{2}\sigma kd[2J\gamma_0+(1-2J)(kd)^2]^{1/2},\quad kd\ll 1. \quad (3.22')$$

To calculate expectation values in $\Psi_0$ in this approximation, we take

$$\langle\alpha_{\mathbf{k}}\alpha_{\mathbf{l}}\rangle=\langle\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{l}}^\dagger\rangle=\langle\alpha_{\mathbf{k}}^\dagger\alpha_{\mathbf{l}}\rangle=0, \quad (3.23\text{a})$$

whereas, from (3.19)

$$\langle\alpha_{\mathbf{k}}\alpha_{\mathbf{l}}^\dagger\rangle=\sigma\delta_{\mathbf{k},\mathbf{l}}. \quad (3.23\text{b})$$

Then,

$$\langle C_{\mathbf{k}}^\dagger C_{\mathbf{k}}\rangle=\tfrac{1}{2}\sigma(\cosh\chi_{\mathbf{k}}-1),\quad \langle C_{\mathbf{k}}^\dagger C_{-\mathbf{k}}^\dagger\rangle=\langle C_{-\mathbf{k}}C_{\mathbf{k}}\rangle=\tfrac{1}{2}\sigma\sinh\chi_{\mathbf{k}}. \quad (3.24)$$

Using (3.15), the correction to the first-order perturbation result (2.11) for the ground-state energy is

$$\Delta E_0=E_0-\tfrac{1}{2}\gamma_0AN\rho_0\approx\tfrac{1}{2}\textstyle\sum_{\mathbf{k}}(\epsilon_{\mathbf{k}}-\tfrac{1}{2}\sigma\gamma_0), \quad (3.25)$$

where we have used (1.18′). This correction is inherently negative for all positive $A$. From (3.17) and (3.24), we have

$$M\textstyle\sum_\delta\langle\sigma_j^x\sigma_{j+\delta}^x\rangle\approx\sigma\sum_{\mathbf{k}}\gamma_{\mathbf{k}}(\cosh\chi_{\mathbf{k}}+\sinh\chi_{\mathbf{k}}) =4\sigma\dfrac{d}{dJ}\left(\dfrac{\Delta E_0}{\sigma}\right). \quad (3.26)$$

The last step results from differentiating (3.25) and using the diagonalization condition (3.20). The other terms in (3.7) and (3.10) are of higher order and cannot be evaluated correctly in the approximation using (3.16). Then, in this approximation,

$$M\mu \approx M\mu_0 + M(\cos\theta/2\sigma)\sum_\delta\langle\sigma_j^x\sigma_{j+\delta}^x\rangle = MA\gamma_0\rho_0 + 2A\cos\theta\frac{d}{dJ}\left(\frac{\Delta E_0}{\sigma}\right);$$

$$\langle H_1\rangle \approx \tfrac{1}{2}M\gamma_0\rho_0^2 + \frac{\sigma\sin^2\theta}{2}\frac{d}{dJ}\left(\frac{\Delta E_0}{\sigma}\right).$$

Thus, (3.8) and (3.9) are consistent with each other and with (3.25) in this approximation to (and only to) the order in which the deviation of $\sigma$ from unity can be neglected, since, for $\sigma=1$, we have from (3.4) and (3.14),

$$d\rho_0 = \tfrac{1}{2}\sin\theta d\theta,$$
$$dJ = \tfrac{1}{2}\sin^2\theta dA + 2A\cos\theta d\rho_0.$$

From (3.3) and (3.24), $\sigma$ is determined by

$$\sigma^{-1}(J) = M^{-1}\sum_{\mathbf{k}}\cosh\chi_{\mathbf{k}}. \qquad (3.27)$$

For $kd \ll 1$, $\cosh\chi_{\mathbf{k}}$ behaves as $k^{-1}$ and, consequently, for a finite system (3.27) diverges and $\sigma=0$. This is due to the infinite fluctuations of the "spin wave" with $\mathbf{k}=0$ and means that our treatment cannot be used, as it stands, for a finite system. A similar situation arises in the spin-wave treatment of antiferromagnetism.[15] However, in the limit $\Omega\rightarrow\infty$, the $\mathbf{k}$-space summation may be replaced by an integral which converges in (two and) three dimensions (but diverges logarithmically in one dimension) because of the $k$-space volume element. In Appendix I it is shown that for values of $J$ in the range in which we are interested, $0\leq J\leq 1$, $\sigma(J)$ satisfies

$$1 \leq \sigma^{-1} \leq 1.156, \qquad (3.28)$$

where the lower (upper) limit of $\sigma^{-1}$ corresponds to the lower (upper) limit of $J$.

To ascertain the behavior of $\sigma$ near $J=0$, $\cosh\chi_{\mathbf{k}}$ can be expanded in a Taylor series in $J$ and integrated term by term everywhere except in the immediate neighborhood of the $k$-space origin where a different procedure is needed as is shown in Appendix I. One gets

$$\sigma^{-1} = (2\sqrt{3}/\pi^2)J^{3/2}(1+2J)(1-2J)^{-2} + \sum_{n=0}^{\infty}(a_n/n!)J^n, \quad a_0=1, \quad a_1=0, \qquad (3.29)$$

the half-integral power dependence being the contribution from the $k$-space origin. Then, to lowest order in $J$,

$$\sigma \approx 1-(2\sqrt{3}/\pi^2)J^{3/2}. \qquad (3.29')$$

[15] P. W. Anderson, Phys. Rev. **86**, 694 (1952).

## Ground-State Energy

To determine the correction to the ground-state energy for small $J$ we first write (3.25) in the form

$$\Delta E_0/M = (\gamma_0/4)\sigma F(J), \qquad (3.25')$$

with an obvious definition for $F(J)$. After replacing the summation by integration, the behavior of $F$ for small $J$ is calculated by the same procedure as was used for $\sigma$. (See Appendix I.) The expansion for $F$ has the form

$$F(J) = (16\sqrt{3}/5\pi^2)J^{5/2}(1-2J)^{-2} + \sum_{n=0}^{\infty}(C_n/n!)J^n,$$
$$C_0 = C_1 = 0; \quad C_2 \approx -0.516, \qquad (3.30)$$

where, again, the half-integral power dependence is the contribution from the $k$-space origin. By keeping terms in $F$ through order $J^{5/2}$, the ground-state energy as calculated from (3.25′) will be consistent with (3.8) and (3.9) through order $A^{5/2}$, with $\sigma=1$. Thus,

$$F(J) \approx -\tfrac{1}{2}|C_2|J^2 + (16\sqrt{3}/5\pi^2)J^{5/2}. \qquad (3.30')$$

Combining (3.4) and (3.14) we see that

$$J \approx 2A\rho d^3(1-\rho d^3), \qquad (3.31)$$

to the order to which we are working ($\sigma\approx 1$). Finally, combining (3.25), (3.30′), and (3.31), and going over to ordinary energy units (i.e., multiplying by $\hbar^2/md^2$), we have, to order $A^{5/2}$,

$$E_0/N \approx (3\hbar^2A\rho d/m)\{1-|C_2|A(1-\rho d^3)^2 + (16\sqrt{3}/5\pi^2)(2A)^{3/2}(\rho d^3)^{1/2}(1-\rho d^3)^{5/2}\}. \qquad (3.32)$$

The expression (3.32) for the ground-state energy may be cast into a more illuminating form which permits comparison with other treatments, by expressing the unphysical model parameters $A$ and $d$ in terms of the zero-energy two-particle scattering length $f_0$ and the depletion parameter $\xi_0$, Eq. (2.14), of the "Bose condensate" in the unperturbed ground state. $f_0$ can be calculated by the method used by Dyson to treat the scattering of two spin waves in the isotropic case.[8] This is done in Appendix II. The (exact) result is

$$2\pi f_0 = 3Ad(1+|C_2|A)^{-1}, \qquad (3.33)$$

where $|C_2|\approx 0.516$ is the same numerical coefficient occurring in (3.30). Using (3.33) and

$$\xi_0 = \rho d^3, \qquad (2.14)$$

we have, to order $f_0^{5/2}$,

$$E_0/N = (2\pi\hbar^2\rho f_0/m)\{1+(128/15\pi^{1/2})(\rho f_0^3)^{1/2}(1-\xi_0)^{5/2} + (4\pi|C_2|/3)(\rho f_0^3)^{1/3}\xi_0^{2/3}(1-\tfrac{1}{2}\xi_0)\}. \qquad (3.34)$$

## Discussion

In the low-density limit, where $\xi_0\rightarrow 0$, Eq. (3.34), to the order of its validity, agrees exactly with the well-known low-density expansion[4,5,7] in terms of $\rho f_0^3$. However, the last term ($\sim f_0^2\xi_0^{2/3}$) in (3.34) represents a higher density correction which has not appeared in other treatments. The nature of this correction, which originates from the $A^2$ term in (3.32), can be understood as follows:

The $A^2$ term represents the second-order perturbation energy $W_2$ of the reduced Hamiltonian (3.15) acting on the unperturbed quasiparticle vacuum $|\theta\rangle$. The intermediate states are $C_{\mathbf{k}}^\dagger C_{-\mathbf{k}}^\dagger|\theta\rangle$, so that

$$W_2 = -\tfrac{1}{2}\sum_{\mathbf{k}}'(\tfrac{1}{2}J\gamma_{\mathbf{k}})^2(2\epsilon_{\mathbf{k}}^0)^{-1} \rightarrow -(J^2/8)\Omega(2\pi)^{-3} \times\int\int\int \gamma_{\mathbf{k}}^2(\gamma_0-\gamma_{\mathbf{k}})^{-1}d^3k = -(J^2/8)M\gamma_0|C_2|,$$

[see Eq. (A6), Appendix I]. In ordinary energy units, using (3.31) and (2.14), we thus have

$$W_2/N = -3\hbar^2A^2\rho d|C_2|(1-\xi_0)^2/m. \tag{3.35}$$

This is just the $A^2$ term in (3.32). Similarly the term proportional to $A$ in (3.32) is, of course, the first-order perturbation energy $W_1/N$, Eq. (2.11). From (3.33) we have, to order $A^2$,

$$2\pi\hbar^2\rho f_0/m \approx (3\hbar^2\rho d/m)A(1-|C_2|A),$$

which is $(W_1+W_2)/N$ for $\xi_0=0$.

Thus, in the limit $\xi_0 \rightarrow 0$, no second-order term appears in the expansion of $E_0/N$ in terms of $f_0$. This "cancellation" of the second-order perturbation contribution is a well-known feature of the pseudopotential[4] and $t$-matrix[5] treatments of the dilute hard-core Bose system.[11] It is clear, however, that this cancellation can be complete only if the unperturbed many-body ground state is an independent-particle one, that is, if the perturbation represents the total interaction, since $f_0$ is a two-particle property, whereas $W_2$ involves two-particle excitations from a many-body ground state.

In the present model the unperturbed ground state $|\theta\rangle$ approaches the free-particle ground state only in the limit $\rho \rightarrow 0$. The many-body effects of the interaction in the unperturbed system ($f_0=A=0$) are expressed by the Bose condensation depletion parameter $\xi_0$ which, in the present model, is proportional to $\rho$ and represents an excluded-volume effect of the hard-core constraint. In other treatments[4,7] the depletion is itself a perturbation effect, proportional in lowest order to $(\rho f_0^3)^{1/2}$. In this connection it should be pointed out that the perturbation does produce a further depletion of the Bose condensate in the present model. One finds, to order $f_0^{3/2}$,

$$\xi = 1-\langle\nu_0\rangle/N \approx \xi_0 + (8/3\pi^{1/2})(\rho f_0^3)^{1/2}(1-\xi_0)^{3/2}. \tag{3.36}$$

In the low-density limit, where $\xi_0 \rightarrow 0$, this agrees with the result of Lee, Huang, and Yang.[4]

The exact form of the term $\sim\rho f_0(\rho f_0^3)^{1/3}\xi_0^{2/3}$ in the ground-state energy is, of course, determined by the specific properties of the pseudospin model. It would seem, however, that a contribution of this kind should be present in general for a Bose system in which the hard-core repulsion plus a longer range attraction combine to produce a small positive scattering length but strong many-body effects.

It is perhaps surprising that the crude pseudospin model, in spite of its obvious unphysical features, should in the simple random-phase approximation reproduce exactly the well-known low-density limit result for the ground-state energy of the hard-core Bose system. This, together with the fact that the model, again in a simple approximation, is able to describe rather well at least *some* features of the λ transition in liquid helium,[1] gives rise to the hope that refinement of the approach based on the kinematical treatment of the hard core may lead to a feasible method of treating the liquid $He^4$ problem.

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## APPENDIX I

Here we show how expressions (3.29′) for $\sigma$ and (3.30′) for $F(J)$ are obtained. By using (3.21) for $\cosh\chi_{\mathbf{k}}$, replacing the $k$-space sum by an integral, and defining

$$u,v,w = k_xd, k_yd, k_zd, \qquad \Gamma = \gamma_{\mathbf{k}}/\gamma_0, \tag{A1}$$

(3.27) becomes

$$\sigma^{-1}(J) = (2\pi)^{-3}\int\int\int_{-\pi}^{\pi}(1-\Gamma+J\Gamma) \times[(1-\Gamma)^2+2J\Gamma(1-\Gamma)]^{-1/2}\,dudvdw. \tag{A2}$$

By the same procedure $F(J)$ in (3.25′) becomes

$$F(J) = (2\pi)^{-3}\int\int\int_{-\pi}^{\pi}\{[(1-\Gamma)^2+2J\Gamma(1-\Gamma)]^{1/2}-1\} \times dudvdw. \tag{A3}$$

Consider (A2) first. For values of $J$ in the range $0\leq J\leq 1$ limits for the magnitude of $\sigma$ are easily established. Since the hyperbolic cosine is $\geq 1$ for all $\mathbf{k}$, $\sigma^{-1}\geq 1$ with equality being achieved at $J=0$. To determine the upper limit the integrand of (A2) can be rewritten as $f(\Gamma)(1-\Gamma^2)^{-1/2}$ with

$$f(\Gamma) = (1-\Gamma+J\Gamma)(1+\Gamma)^{1/2}(1-\Gamma+2J\Gamma)^{-1/2}.$$

Since $-1\leq\Gamma\leq 1$ and $0\leq J\leq 1$, one sees that $0\leq f(\Gamma)\leq 1$. It follows that

$$\sigma^{-1}(J) \leq (2\pi)^{-3}\int\int\int_{-\pi}^{\pi}(1-\Gamma^2)^{-1/2}\,dudvdw,$$

or

$$\sigma^{-1}(J)\leq\sigma^{-1}(1)=1.156, \quad (A4)$$

where we have used Anderson's[15] evaluation of the integral $\sigma^{-1}(1)$. Hence, (3.28) follows.

To find the behavior of $\sigma(J)$ near $J=0$, the integrand in (A2) can be expanded into a Taylor series in $J$ and integrated term by term everywhere except in the immediate vicinity of the $k$-space origin, where $\Gamma\rightarrow 1$. But in that region one can certainly use the isotropic expression for $\Gamma$,

$$\Gamma\approx 1-r^2/6,\quad r^2=u^2+v^2+w^2,\quad r\ll 1.$$

An integral over a small sphere of radius $r_0$ about the origin then takes the form

$$(2\pi^2\eta^{1/2})^{-1}\int_0^{r_0}[r^2(1-J)+6J][r^2+12J/\eta]^{-1/2}\,dr,$$

with $\eta=1-2J$. This integrates to

$$(4\pi^2\eta^{1/2})^{-1}\{(r^2+12J/\eta)^{1/2}\times[12J-16J(1-J)/\eta+\tfrac{2}{3}(1-J)r^2]\}_0^{r_0}.$$

Thus, the contribution from the origin is

$$2\sqrt{3}\pi^{-2}J^{3/2}(1+2J)(1-2J)^{-2}. \quad (A5)$$

For 12 $J/\eta<r_0^2$ the contribution from the upper limit can be absorbed into the Taylor series. Hence, (3.29) follows. One finds for the first two coefficients of the power series, $a_0=1$, $a_1=0$. Then, to order $J^{3/2}$, (3.29′) results.

Following the same procedure used for $\sigma$, the behavior of (A3) near $J=0$ can be ascertained. The contribution from the $k$-space origin is $(16\sqrt{3}/5\pi^2)J^{5/2}(1-2J)^{-2}$. The first nonzero coefficient of the Taylor series is

$$C_2=(d^2F/dJ^2)_{J=0}=-(2\pi)^{-3}\times\int\int\int_{-\pi}^{\pi}\Gamma(1-\Gamma)^{-1}\,du\,dv\,dw=-0.516, \quad (A6)$$

where the evaluation of the integral is due to Watson.[16] Hence, to order $J^{5/2}$, (3.30′) follows.

## APPENDIX II

To evaluate the zero-energy scattering length for two-particle collisions we use the method of Dyson[8] in his general theory of spin-wave interactions. We immediately go over from the Hamiltonian (1.17) to the effective (non-Hermitian) Bose Hamiltonian for ideal spin waves:

$$H_{\text{Bose}}=\sum_{\mathbf{k}}\epsilon_{\mathbf{k}}{}^0\beta_{\mathbf{k}}{}^\dagger\beta_{\mathbf{k}}-(4M)^{-1}\sum_{\mathbf{klm}}(\Gamma_{\mathbf{lm}}{}^{\mathbf{k}}-2A\gamma_{\mathbf{k}})\times\beta_{\mathbf{l-k}}{}^\dagger\beta_{\mathbf{m+k}}{}^\dagger\beta_{\mathbf{l}}\beta_{\mathbf{m}}\equiv H_1+H_2, \quad (A7)$$

[16] G. N. Watson, Quart. J. Mech. Appl. Math. **10**, 266 (1939).

where the notation is generally the same as Dyson's except that we use $\beta^\dagger,\beta$ for the creation, destruction operators which obey ordinary Bose commutation rules. The difference between Dyson's Eq. (48) and (A7) is the presence of the anisotropy term. In terms of the local Bose operators we have

$$H_1=\tfrac{1}{4}\sum_{j,\delta}(\eta_j{}^\dagger-\eta_{j+\delta}{}^\dagger)(\eta_j-\eta_{j+\delta}),$$
$$H_2=\tfrac{1}{4}\sum_{j,\delta}\eta_j{}^\dagger\eta_{j+\delta}{}^\dagger(\eta_j-\eta_{j+\delta})^2+\tfrac{1}{2}A\sum_{j,\delta}\eta_j{}^\dagger\eta_{j+\delta}{}^\dagger\eta_{j+\delta}\eta_j. \quad (A8)$$

A state containing two noninteracting particles with zero center-of-mass momentum can be written as

$$|\Psi_{\text{in}})=\sum_{ij}\cos[\boldsymbol{\mu}\cdot(\mathbf{r}_i-\mathbf{r}_j)]\eta_i{}^\dagger\eta_j{}^\dagger|0), \quad (A9)$$

where $\boldsymbol{\mu}$ is the wave vector for the relative motion of the two particles and $|0)$ is the vacuum state for the $\eta_j$ operators. Following Dyson, we construct a state $|\Psi)$ representing the same two particles with interaction, satisfying the Schrödinger equation

$$(H_1+H_2)|\Psi)=2\epsilon_\mu{}^0|\Psi). \quad (A10)$$

A Green's function $G(\mathbf{r}_i-\mathbf{r}_j)$ for $H_1$ is defined by

$$(H_1-2\epsilon_\mu{}^0)G(\mathbf{r}_i-\mathbf{r}_j)=\delta(\mathbf{r}_i-\mathbf{r}_j), \quad (A11)$$

which, when solved for $G$ gives

$$G(\mathbf{r}_i-\mathbf{r}_j)=M^{-1}\sum_{\mathbf{k}}\exp[i\mathbf{k}\cdot(\mathbf{r}_i-\mathbf{r}_j)]\cdot[2(\epsilon_{\mathbf{k}}{}^0-\epsilon_\mu{}^0)]^{-1}. \quad (A12)$$

When the separation $r=|\mathbf{r}_j-\mathbf{r}_i|$ is large, $G$ takes the asymptotic form

$$G(r)\sim d\exp(i\boldsymbol{\mu}\cdot\mathbf{r})/4\pi r. \quad (A12')$$

For the state $|\Psi)$ satisfying (A10) we take

$$|\Psi)=\sum_{ij}\{\cos[\boldsymbol{\mu}\cdot(\mathbf{r}_i-\mathbf{r}_j)]+\sum_\delta B_\delta G(\mathbf{r}_j-\mathbf{r}_i-\boldsymbol{\delta})\}\eta_i{}^\dagger\eta_j{}^\dagger|0), \quad (A13)$$

which has the asymptotic form appropriate for an incident plus scattered wave:

$$|\Psi)\sim\sum_{i,j}\{\cos[\mu\cdot(r_i-r_j)]+(de^{i\mu r}/4\pi r)\sum_\delta B_\delta\}\eta_i{}^\dagger\eta_j{}^\dagger|0). \quad (A13')$$

The coefficients $B_\delta$, of course, will differ from those in Dyson's paper because of the presence of the anisotropy term in (A7).

By virtue of (A11), the Schrödinger equation (A10) gives an equation to determine the $B_\delta$:

$$B_\delta=(1-A)\cos\boldsymbol{\mu}\cdot\boldsymbol{\delta}-1+\sum_\Delta B_\Delta\times\{(1-A)[G(\boldsymbol{\Delta}-\boldsymbol{\delta})+G(\boldsymbol{\Delta}+\boldsymbol{\delta})]-G(\boldsymbol{\Delta})\}$$
$$=(1-A)\cos(\boldsymbol{\mu}\cdot\boldsymbol{\delta})-1-M^{-1}\sum_\Delta B_\Delta\sum_{\mathbf{k}}\exp(i\mathbf{k}\cdot\boldsymbol{\Delta})\times(\gamma_\mu-\gamma_{\mathbf{k}})^{-1}[A\cos(\mathbf{k}\cdot\boldsymbol{\delta})+(1-\cos\mathbf{k}\cdot\boldsymbol{\delta})], \quad (A14)$$

where $\boldsymbol{\Delta}$ is a nearest-neighbor lattice vector. Since we are interested in the scattering length at zero energy, we take the limit of (A14) as $\boldsymbol{\mu}\rightarrow 0$ and use the fact that in this limit the $B_\delta$ cannot depend upon the direction of $\boldsymbol{\delta}$. Thus, one finds that

$$\sum_\delta B_\delta = -\gamma_0 A/[1+A(\gamma_0 M)^{-1} \times\sum_{\mathbf{k}}\gamma_{\mathbf{k}}^2(\gamma_0-\gamma_{\mathbf{k}})^{-1}]. \quad (A15)$$

Replacing the sum over $\mathbf{k}$ by an integral and remembering (1.18′), this becomes

$$\sum_\delta B_\delta = -\gamma_0 A/[1+A|C_2|], \quad (A15')$$

where

$$-C_2=\frac{1}{(2\pi)^3}\int\int\int\Gamma(1-\Gamma)^{-1}\,du\,dv\,dw=0.516, \quad (A6)$$

as in Appendix I. Inserting (A15′) into the asymptotic wave function, one obtains for the scattering length at zero energy,

$$f_0=A\gamma_0 d[4\pi(1+A|C_2|)]^{-1} =3Ad[2\pi(1+A|C_2|)]^{-1}, \quad (A16)$$

which is the expression given in the text, Eq. (3.33).

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# Magnetoacoustic Effects in Longitudinal Fields*

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The absorption of transverse and longitudinal ultrasonic waves has been studied as a function of the magnetic field applied along the direction of propagation in pure tin and lead crystals at liquid-helium temperatures. At low fields, the attenuation of both transverse and longitudinal waves exhibits oscillations approximately periodic in reciprocal field, which are ascribed to electron orbits which execute a periodic motion along the field direction. The attenuation of transverse waves in tin shows regions of rapid decrease with magnetic field, which are interpreted as the absorption edges first predicted by Kjeldaas. In higher fields, the absorption of longitudinal waves appears to saturate at a nonzero value, while that of the transverse waves in tin appears generally still to be decreasing with field at fields of about 10 kG.

## 1. INTRODUCTION

A LARGE number of experiments have been carried out on the effect of a transverse magnetic field on the absorption of ultrasonic waves in pure metals at low temperatures, and the study of the angular variation of the magnetoacoustic oscillations has contributed significantly to the knowledge of the Fermi surfaces of many metals. Few results have so far been obtained on the dependence of the attenuation on a longitudinal magnetic field, however, partly because the results cannot be so readily interpreted in terms of the geometrical parameters of the Fermi surface, and partly on account of the experimental difficulties involved. These measurements do have a certain intrinsic interest, however, and the present work represents an attempt to understand the coupling between acoustic waves and the conduction electrons in metals in the presence of a longitudinal magnetic field, while simultaneously obtaining some information about the Fermi surfaces of the metals studied.

In the following sections the experimental technique used in these measurements is described briefly and the experimental results are presented. The theory of the attenuation of ultrasonic waves in longitudinal magnetic fields is then discussed, and finally the results are interpreted in the light of this theory. A brief account of some of the results of this work has already been published.[1]

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## 2. EXPERIMENTAL METHOD

The absorption of 80-Mc/sec transverse and longitudinal ultrasonic waves in pure lead and tin crystals in a longitudinal magnetic field was measured by means of a "pulse-echo" technique, the details of which have been described elsewhere.[2]

The longitudinal and shear waves were generated by applying a high-frequency electromagnetic pulse across $X$- and $Y$-cut quartz crystals, respectively, exciting them on their fifth harmonic. Ultrasonic reflections from the free end of the specimen were reconverted by the transducer into electrical signals which were amplified, demodulated, and displayed on a cathode-ray oscilloscope. In practice, because of the high attenuation in the pure crystals used in these experiments, only one reflection could be observed, in the normal state. The

[1] A. R. Mackintosh, in *Proceedings of the Seventh International Conference on Low-Temperature Physics* (University of Toronto Press, Toronto, 1960), p. 12.
[2] A. R. Mackintosh, Proc. Roy. Soc. (London) **A271**, 88 (1963).